6.A: the ***PhilosophyIdeas*** overview of

The Nature of Mathematics

For early philosophers seeking a general understanding of nature, mathematics immediately looked like a tantalising example, because it appeared that it was eternally true (whereas physical reality seemed to fluctuate), that we could be certain about the simpler parts of it, and that we could know it directly by thought (rather than experience). Since it appeared that aspects of nature (such as musical harmony) conformed to mathematics, it might reveal deep truths about nature (as we would say, 'a priori synthetic necessary truths'), or even be the actual foundations of reality. Hence understanding the general nature of mathematics has been an enduring target in philosophy.

Mathematics originates in **counting** and in **measuring**, which gave rise to arithmetic and geometry. The subsequent story has been of continually higher levels of generalisation and abstraction, and the development of new techniques to deal with further puzzles. Although much mathematics has moved beyond numbers, they remain the starting point for philosophers. Most mathematics can be expressed in set theory, and category theory is an even more generalised approach to the structures. Two large questions hang over mathematics: can a unified account be given of the whole subject, and can all of its problems be solved?

A striking feature of mathematics is that the physical world seems to conform to most of its truths, and this influences our view of the nature of the subject. Three plausible interpretations are that maths fits nature because it is derived from it, or because we have specifically constructed maths to model nature, or because maths is so general that it coincides with the most general facts about nature. These roughly fit the Empiricist, Constructivist and Platonist approaches. If maths is **derived from nature**, then the patterns in numbers and geometry are facts external to the subject, and the feeling of necessity in maths reflects the enduring character of the world. Departures from that physical basis are allowable, but trivial. If maths is a human **construction**, motivated by a need to organise nature, then its roots will again reflect nature, but the extensions beyond nature will be just as valid and interesting. If the **generality** of maths is the explanation of its coincidence with nature, then this will make generalisation the main feature of the subject, and the drive for broader generalisations will be explained. For example, the introduction of variables to stand for numbers reveals more general truths than those found in the original arithmetic.

We can approach the nature of numbers by considering counting and measurement, which both fall into the category of '**quantity**'. Not all quantity is numerical, but even vague quantities such as pain or happiness can be quantised (out of 10, or 100). At first it looked as if measurement could be a form of counting, by introducing a 'unit' of size, such as centimetres. Fine-grained quantities could be expressed as ratios (or fractions) of the counting numbers (such as two-fifths of a centimetres). The discovery that the diagonal of a square could not be expressed as a ratio, and the realisation that there was no limit to being fine-grained, led to the discovered of **real** numbers, as well as **natural** (counting) numbers. The real numbers are those with infinite decimal expansions (such as π). These numbers are so numerous that we cannot name them all, and they are beyond our clear understanding. They still seem to be numbers, since they can be summed or rounded up, just like the fractions. Subsequent explorations have given us **complex** numbers, **imaginary** numbers, and a whole science of 'number theory'.

A small but dramatic development, which departed from counting and measurement, was the invention of '**zero**'. Once you have introduced subtraction, and the convenient **negative** numbers, it is obvious that some calculations (such as 3+8-2+5-14) lead to an absence of numbers, which terminates the calculation. Accountants found this frustrating, so the 'absence' was treated as a number in lengthy calculations. Nowadays it is commonplace to say the natural numbers are '0,1,2,3,4…', and even use zero in counting, when different types of thing are being counted. This is 'dramatic' because the number system now has priority over the world it refers to, and new concepts can be introduced to benefit mathematics itself, rather than practical life. The subject now has a life of its own.

If natural numbers handle quantity, then the number '**one**' has an odd status. If you count ten sheep you start with one, but 'ten sheep' contains information about quantity, whereas 'one sheep' doesn't seem to, since the idea of quantity doesn't enter into our experience of a sheep (unless we add arithmetic to the experience). The related idea of '**units**' is intriguing too. We count centimetres, but counting all the metres and the centimetres of a length would be confused. However, you can count the sheep, but you can also count the sheep and the goats together. You could even count the sides of a triangle and the moons of Jupiter in a single operation, but what unit are you using?

If we ask how far the sequence of natural numbers extends, the idea of '**infinity**' is obvious to us, and this infinity is now designated by ω (omega), but every element of this infinity can be matched with one of the numbers, so we have a **countable** infinity. The real numbers, however, are vastly more numerous (said nowadays to be the number of subsets of the natural numbers, labelled $\aleph_1$ [aleph-one]), and so they are **uncountable**. The number of subsets of the real numbers gives another infinity beyond that ($\aleph_2$), and an endless **hierarchy** of infinities is revealed, which quite transforms our picture of the world of numbers.

Geometry was originally taken to be a formal description of natural space, but two developments altered this view. First, it was found that the whole subject could be expressed by equations (using variables plotted with co-ordinates), and then it was found that new geometries could be described, by changing the original axioms. The subject always models possible spaces, but it is no longer importantly different from other mathematics.

As long as the focus was on counting and measuring, and then on the numbers themselves, the nature of mathematics at least occupied a restricted area of thought. Eventually, though, when the logic used in mathematics itself became an object of study, so that logic became a part of mathematics, rather than a way of talking about it. This accelerated the abstract and general understanding of mathematics, and strengthened the platonist idea that mathematics is a separate aspect of reality. The result is that order has replaced quantity as the central concept of mathematics, and it is seen as the study of abstract **structures**, rather than the generalised study of nature.